ALGEBRAIC GEOMETRY - END SEMESTER EXAM

Time: 3 hours

Each question is worth 10 marks. Attempt **any 10 problems**. Maximum marks: 100

All our usual conventions apply e.g. k is an algebraically closed field. You may assume that the field is algebraically closed.

Question 1. Find the singular points of the following curves in $\mathbb{A}^2$.

(1) $x^2 = x^4 + y^4$.
(2) $xy = x^2 + y^2$.

Assume char $k \neq 2$.

Question 2. Show that $\mathbb{A}^1$ is not isomorphic to any proper open subset of itself.

Question 3. Show that $\mathbb{P}^m$ is not isomorphic to $\mathbb{P}^n$ when $m \neq n$.

Question 4. Let $Q = V(x^3 - yz^2) \subset \mathbb{P}^2$. Find $K(Q)$ the rational function field of Q.

Question 5. Let $f \in k[x, y, z]$ be a homogeneous polynomial. Let $X = V(f) \subset \mathbb{P}^2$. If for every $P \in X$, at least one of $\dfrac{\partial f}{\partial x}(P)$, $\dfrac{\partial f}{\partial y}(P)$, $\dfrac{\partial f}{\partial z}(P)$ is nonzero then X is a nonsingular variety.

Question 6. Find all the closed subsets of $\mathbb{P}^1$.

Question 7. Consider the elliptic curve $C = V(y^2 - x(x-1)(x-\lambda)) \subset \mathbb{A}^2$. Find the condition on λ for C to be nonsingular everywhere.

Question 8. Consider the rational map $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ given as

$$[x_0 : x_1 : x_2] \mapsto [x_1x_2 : x_2x_0 : x_0x_1]$$

Show that φ is a birational map. Find its rational inverse. Find open sets $U, V \subset \mathbb{P}^2$ such that φ defines an isomorphism $U \cong V$.

Question 9. Show that any two disjoint lines in $\mathbb{P}^2$ will intersect at a unique point.

Question 10. Let $f = x^2 - y^2$ and $g = x^3 + xy^2 - y^3 - x^2y - x + y$. Find ALL the irreducible components of $V(f, g) \subset \mathbb{A}^2(\mathbb{C})$

Question 11. Give an example of a noetherian toplogical space of infinite dimension.